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HEAT AND MASS TRANSFER AND PHYSICAL GASDYNAMICS

The Use of Inverse Method of Solving Boundary-Layer Equations for the Testing of Turbulence Models

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Abstract—Comparison is made of the classical direct method and the so-called inverse method of solving turbulent boundary-layer equations as applied to the problem of testing of turbulence models. In the case of the direct method, it is assumed that the velocity distribution along the external bound of the boundary layer is known from experiment, and, in the case of the inverse method, it is assumed that it is the longitudinal distribution of the displacement thickness that is preassigned from experiment. It is demonstrated that the inverse method (which, unlike the direct one, does not require the preassignment of the longitudinal gradient of velocity at the external bound of the boundary layer) enables one to derive more objective (free of arbitrariness in preassigning the input data) information about the capabilities of the turbulence models being tested. This method is used to perform a detailed investigation of the capabilities of a wide scope of algebraic, semidifferential, and differential turbulence models in calculating boundary layers with adverse, accelerating, and alternating-sign pressure gradients, and conclusions are formulated about the advantages and disadvantages of individual models.

INTRODUCTION

The calibration and testing of semiempirical turbulence models, that is, the determination of the values of empirical constants entering those models and the estimation of their adequacy and ranges of validity by comparing the calculation results with experimental data, are essential stages in the development of new and estimation of the capabilities of the existing models. In performing such investigations, computational errors inherent in any numerical solution, if not fully eliminated, must be reduced to a minimum. The same is true of experimental data, especially, those of them which are used as the boundary conditions or closing relations in performing calculations. Although it is claimed that present-day turbulence models are capable of describing a fairly wide class of turbulent flows (with flow separation and attachment, substantial curvature of the lines of flow, sharp variation of conditions on the surface being flown about, etc.), because of the reasons identified above, a requisite stage in the testing of any turbulence model is the estimation of its capabilities in calculating relatively simple canonical turbulent flows. As applied to wall flows, such a requisite test consists in the calculation of turbulent boundary layer on a solid surface. This is due to the fact that the present-day level of computer development enables one to fairly easily derive rather exact (free of computational errors due to the coarseness of grids) solutions of boundary-layer equations, while for more complex flows, described by two- and, the more so, three-dimensional Reynolds equations, the problem remains very complicated and labor-consuming. Besides, even this "simplest" flow is, from the physical standpoint, fairly informative, because it is characterized by a number of complex effects. For example, an adequate description of the effect of strong longitudinal pressure gradients on the structure and basic characteristics of turbulent boundary layer proves too difficult a problem for many known turbulence models. Therefore, before turning to the estimation of the capabilities of models in calculating complex flows, one must first test them in application to flows of the boundary-layer type, for which extensive and fairly reliable experimental data have been accumulated.

The problem of objective comparison of numerical solutions with experimental data remains topical for flows of the boundary-layer type as well. The reason for this is as follows.

Within the boundary-layer theory, it is assumed that the velocity on the external bound of the boundary layer, the value of which must be known for preassigning the boundary conditions, and its longitudinal gradient entering the equation of motion must be found independently from the solution of Euler's equations for nonviscous flow past the surface being investigated. Given such an approach, the above-identified computational advantages of turbulent layer approximation, as compared with complete Navier–Stokes (Reynolds), equations are reduced considerably, because the solution of Euler's equations *per se* presents a fairly complicated problem. In addition, such an approach, generally speaking, calls for the use of a special iteration pro-

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cedure with respect to the boundary layer displacement thickness [1], enabling one to include the inverse effect of the boundary layer on the nonviscous layer. Along with this, in order to perform the above-described calculation, one needs data on the geometry of the surface being flown about for the entire flow, which, as a rule, are absent in the case of experimental studies involving the measurement of boundary layer characteristics. In view of this, the above-identified procedure is not employed in testing turbulence models within the boundary layer approximation, and the respective values, measured in the experiment, are preassigned as the boundary conditions on the external bound of the boundary layer. The gradient of longitudinal velocity, entering the equation of motion, is determined in just the same way (by the experimental data). The latter fact may lead to considerable uncontrollable calculation errors. This is associated both with the asymptotic behavior of the boundary layer equations (the concept of the finite layer thickness is conventional and hard-to-determine in experiments when the flow in nonviscous flow core is substantially two- or even three-dimensional) and with the need to calculate the derivative of velocity at the external boundary on the longitudinal coordinate dU_e/dx by the experimental points for $U_e(x)$, which entails great errors and an element of subjectivity associated with the use of different methods of "smoothing" the experimental data. Indirectly indicative of the gravity of the above-described difficulties are the results of numerous experiments in which boundary layers with a strong adverse (unfavorable) pressure gradient were investigated (see, for example, [2, 3]) and the Karman momentum integral equation was violated,

$$\frac{d\theta}{dx} + \frac{dU_e/dx}{U_e}(2\theta + \delta^*) = \frac{C_f}{2}, \quad (1)$$

where θ and δ^* denote the integral momentum thickness and the integral displacement thickness of boundary layer, respectively, and C_f is the friction coefficient.

At first glance, this rules out the possibility of agreement between the results of boundary layer calculations with the above-mentioned experimental data, irrespective of the type of turbulence model employed in the calculations, and gives grounds to doubt the validity of the classical equations of turbulent boundary layer. For example, Dmitriev [4] and Tsahalis and Telionis [5] suggest that it is necessary to include the anisotropy of normal Reynolds stresses in the preseparation region of turbulent boundary layer, which leads to the emergence of an additional term in the boundary layer equations (the so-called Van Le equations [6]) and to a respective change in Karman equation (1).

Without disputing this hypothesis, note, however, that if one estimates the error in determining the velocity gradient on the external boundary, which leads to experimentally observed considerable discrepancies of Karman equation (1), it turns out that this error is not all that great and may easily occur in determining the velocity gradient on the external boundary of the layer by the experimental data for the velocity proper.

Therefore, the use of the direct method of solving equation of turbulent boundary layer when testing turbulence models by comparing the calculation results with experiment entails serious "technological" difficulties and may lead to wrong conclusions about the capabilities of the models being treated. These difficulties may, however, be avoided when testing models by using a different approach to the solution of boundary layer equations, which is usually referred to as the inverse method [7].

The inverse method consists essentially in that it is not the velocity on the external bound of the boundary layer and its gradient which are preassigned from experiment, but the longitudinal distribution of the boundary layer displacement thickness, which is measured fairly accurately in the experiments. By doing so, one can avoid the operation (which involves great errors) of determining the velocity gradient on the external boundary by the experimental data, because, within the framework of the inverse method, both the velocity on the external bound of the boundary layer and its gradient are the sought quantities to be determined in the process of calculations. In estimating the turbulence model, these quantities, along with the remaining calculated characteristics of the boundary layer, must be compared with the respective experimental data.

This study is aimed at demonstrating the advantages of the inverse method (compared with the direct one) and using this method for detailed testing of both a series of well-known turbulence models and those proposed relatively recently as applied to boundary layers with adverse, accelerating, and alternating-sign pressure gradients.

FORMULATION OF THE PROBLEM AND CALCULATION METHOD

The set of equations of turbulent boundary layer on a flat or axisymmetric surface in the case of incompressible liquid flow has the form [1]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{r^{-\alpha}}{\rho}\frac{\partial}{\partial y}\left(r^{\alpha}(\mu + \mu_T)\frac{\partial u}{\partial y}\right) + U_e\frac{dU_e}{dx},$$
$$\frac{\partial(r^{\alpha}u)}{\partial x} + \frac{\partial(r^{\alpha}v)}{\partial y} = 0. \quad (2)$$

Here, x, y and u, v denote the longitudinal and transverse coordinates, and velocity components, respectively; ρ is the density; μ and μ_T denote the molecular and turbulent viscosity, respectively; $\alpha = 0$ for plane and $\alpha = 1$ for axisymmetric flow; and r is the distance from the point at hand to the axis of symmetry.

Preassigned on a solid wall as the boundary conditions for the set of equations (2) are the no-slip conditions for the longitudinal component and impermeability conditions for the transverse component of the velocity vector. As already mentioned, when the direct method of solution (2) is used, the experimentally measured distribution of velocity along the external boundary $U_e(x)$ is preassigned as the second boundary condition for its longitudinal component, and the velocity gradient appearing in equation (2) is determined by some approximate method of differentiating this experimental distribution. As a rule, the experiment yields the values of $U_e(x)$ only in separate points that are quite far apart, and this obviously makes the procedure rather inexact and, in addition, strongly dependent on the employed method of "differentiation."

In the inverse method, the integral displacement thickness δ^* or some other quantity (momentum thickness θ, friction coefficient on the wall C_f) is preassigned as the missing boundary condition from experiment. In so doing, the solution procedure is fully analogous to the methods employed to calculate internal flows within the narrow channel approximation, when the longitudinal distribution of pressure is determined from the integral balance of mass using the appropriate modification of the sweep algorithm during numerical integration of the equation of motion [8]. The only difference is that, in solving the boundary layer equation (2) by the inverse method, the preassignment of the flow rate of liquid through the channel is replaced by the preassignment of the longitudinal distribution of the boundary layer displacement thickness δ^*.

The numerical integration of the set of equations (2) was performed using a two-layer implicit finite-difference schema of the first order of accuracy on the longitudinal coordinate and of the second order of accuracy on the transverse coordinate. At every step with respect to x, the system of difference equations is solved using the above-described modification of the sweep method with iterations for nonlinearity.

TURBULENCE MODELS

The turbulence models treated by us include the Cebeci–Smith algebraic model [9] (CS), two so-called semidifferential models (the Johnson–King model [10] (JK) and the Horton model [11] (HO)), two differential models with one equation for turbulent viscosity (the model of A.N. Sekundov and associates [12] (ν_T-92) and the Spalart–Allmaras model [13] (SA)), and three differential models with two equations (the k–ε models of Launder–Sharma [14] (LS) and Chien [15] (CH) and the k–ω model of Menter [16] (M-SST)).

The algebraic and semidifferential models are constructed according to the classical two-layer Clauser scheme of turbulent boundary layer and employ, as linear and velocity scales defining the turbulent viscosity, quantities characteristic of flows of the boundary layer type, such as the boundary layer thickness δ, the displacement thickness δ^*, the velocity on the external bound of the boundary layer U_e, and the dynamic velocity $\vartheta_* = (\tau_w/\rho)^{1/2}$, which makes difficult (and sometimes rules out) the possibility of using them to calculate flows of complicated geometry. From this standpoint, the differential models of [12–16] are more universal.

All of the models listed above, except for the Horton model [11], are well known and described in detail in a number of papers. Therefore, we will formulate only the basic relations of the Horton model, according to which the turbulent viscosity is found by the formula

$$\nu_T = \nu_{TO}\tanh(\nu_{TI}/\nu_{TO})[1 + (0.17y/\delta^*)^{6.75}]^{-1}. \quad (3)$$

Here, the quantities ν_{TI}, ν_{TO}, p^+, and the damping factor D are defined by the relations

$$\nu_{TI} = (\kappa y D)^2 \frac{\tau}{\tau_w}\left|\frac{\partial u}{\partial y}\right|, \nu_{TO} = \sigma(x)kU_e\delta^*,$$

$$D = 1 - \exp\left(-\frac{y v_*}{\nu}\frac{(1 - bp^+)^{1/2}}{A}\right),$$

$$p^+ = \frac{\nu U_e}{v_*^3}\frac{\partial U_e}{\partial x},$$

and the parameter σ is found from the solution of the following ordinary differential equation:

$$\delta\frac{d(as)}{dx} = C_1 s(1 - a),$$

where δ is the boundary layer thickness determined by the level of $u/U_e = 0.995$, $a = \sigma^{0.69}$, and the quantity $s(x)$ is determined by the formula

$$s = \left(\frac{\delta^*}{U_e}\frac{\partial u}{\partial y}\bigg|_{y = 0.5\delta}\right)^{0.69}.$$

The model constants are as follows: $\kappa = 0.41$, $A = 26$, $k = 0.018$,

$$b = \begin{cases} 14.76 \text{ for } p^+ \geq 0 \\ 12.60 \text{ for } p^+ < 0, \end{cases} \quad C_1 = \begin{cases} 0.5 \text{ for } \sigma > 0 \\ 0.14 \text{ for } \sigma \leq 0. \end{cases}$$

COMPARISON OF THE DIRECT AND INVERSE METHODS

In order to compare the capabilities of the direct and inverse methods of solving boundary layer equations in testing turbulence models, two typical flows in boundary layers in the presence of an adverse pressure gradient were selected, which were experimentally investigated by Ludvig–Tielman (see the Stanford Conference Proceedings [2]) and Dangel-Fernholz [17]. In the first instance (Experiment 1200), the flow near a plane surface was treated, and in the second instance, the flow near an axisymmetric surface. The calculations of these two flows were performed by both direct and inverse

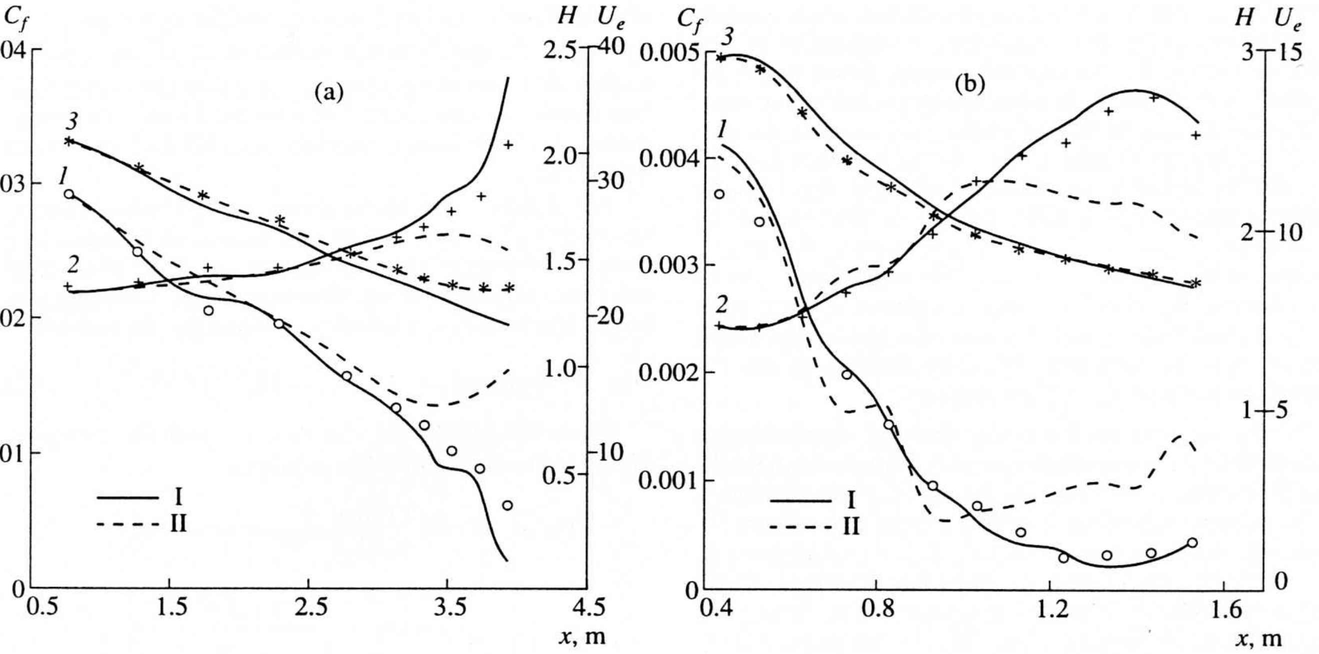

Fig. 1. Comparison of the direct and inverse methods of solving boundary layer equations when using the M-SST model, as illustrated by the example of experiments (a) 1200 and (b) DF: I, inverse method; II, direct method; (*1*) longitudinal distribution of the friction coefficient C_f, (*2*) longitudinal distribution of the form parameter H, (*3*) longitudinal distribution of the velocity U_e on the external bound of the boundary layer. The curves indicate calculation results, and the points indicate experimental data.

methods, using the M-SST model, which, according to the available literature data, provides a fairly accurate description of boundary layers with adverse pressure gradient.

The flow parameters defining the flow ($U_e(x)$ for the direct method and $\delta^*(x)$ for the inverse one) were preassigned from experiment without any smoothing, and the experimental data were interpolated to the computational grid with the aid of cubic splines. In so doing, the derivative of velocity on the external bound of the boundary layer (which must be preassigned in the case of direct method) was determined analytically by spline differentiation.

The calculation results given in Fig. 1 illustrate fairly clearly the nature and scale of the differences between the direct and inverse approaches to the solution of boundary layer equations. In particular, an analysis of these results produces diametrically opposite conclusions about the capabilities of the turbulence model being treated. According to the direct method, the M-SST model is incapable of correctly predicting the characteristics of a boundary layer on approaching the separation point, while the results derived with the aid of this model by the inverse method (including the longitudinal distribution of velocity) are in adequate agreement with the experimental data in the entire region of flow. Note that the selected method of "restoring" the velocity gradient on the external bound of the boundary layer by experimental data (using splines) is not the only available, and other methods may be apparently used to considerably improve the agreement of the results of calculation by direct method with the experimental results for the flows treated above. However, the foregoing examples clearly indicate that no such problem arises in the case of the inverse method, and this obviously makes the latter method a more objective tool for testing turbulence models. Given below are the results of such testing for a fairly wide class of turbulent boundary layers with pressure gradient.

RESULTS OF TESTING OF TURBULENCE MODELS

In order to estimate the capabilities of eight turbulence models listed above, we selected a series of experimental investigations of plane and axisymmetric boundary layers, which yielded fairly complete and reliable results of measurements of the main characteristics of flow in a wide range of variation of parameters on the external boundary of the layer. The list of treated flows with their brief characteristics is given in the table. A part of the experimental data were borrowed from the Stanford Conference Proceedings [2] (in the table, the respective experiments are given the same designations as in [2]). The remaining experimental data were borrowed from original papers, references to which are also made in the table.

The calculation results are given in Figs. 2–10.

Table

Designation	Types of flow	Refs.
2700	Accelerating pressure gradient	Herring, Norbury [2]
0141	Adverse pressure gradient	Samuel, Joubert [18]
3300	Adverse pressure gradient	Bradshaw [2]
4800	Adverse pressure gradient	Schubauer, Spangenberg [2]
TM	Alternating-sign pressure gradient	Tsuji, Morikawa [19]
1200	Adverse pressure gradient (preseparation flow)	Ludweig, Tillman [2]
0431	Alternating-sign pressure gradient (preseparation flow)	Simpson, Chew, Shivaprasad [3]
DF	Adverse pressure gradient (axisymmetric preseparation flow)	Dangel, Fernholz [17]

As was to be expected, the results of calculation of flow with a favorable pressure gradient (experiment 2700) are little sensitive to the turbulence model employed and, by and large, agree fairly well with the experimental data (see Fig. 2). The greatest discrepancy between the models is observed in the case of longitudinal distribution of the friction coefficient. The best agreement with experiment with respect to this parameter is provided by the simplest of algebraic and semidifferential models, namely, the CS model (see Fig. 2a). Both differential models with one equation (ν_T-92 and SA models) yield results that are very close to one another (only an insignificant advantage of the ν_T-92 may be noted) and exceed considerably all models in the first group as regards the accuracy of calculation of friction (see Figs. 2a, 2b). Of the differential models with two equations (see Fig. 2c), the M-SST model is advantageous over the models of the k–ε group (LS and CH models) and quite comparable in accuracy with the ν_T-92 model (see Figs. 2b, 2c). The situation is different when calculating boundary layers with an adverse (unfavorable) pressure gradient. Before turning to the analysis of the results, it is to be reminded that, as already mentioned in the introduction, most of the experimental results exhibit more or less significant violation of the Karman momentum integral equation (1), and it is for this reason that the data of experiments 1200 and 0431 have not been used recently to test turbulence models (see, for example, [20]). The results given below indicate that, when the inverse method of solving boundary layer equations is used, the above-identified "defect" of these experiments does not play a significant role and manifests itself only as some mismatch of the calculated and experimentally obtained values of velocity distribution on the external bound of the boundary layer, which may be quite possibly attributed to the reasons specified at the beginning of this paper.

The calculation results pertaining to boundary layers with an adverse pressure gradient are given in Figs. 3–7. Their analysis leads to the following conclusions.

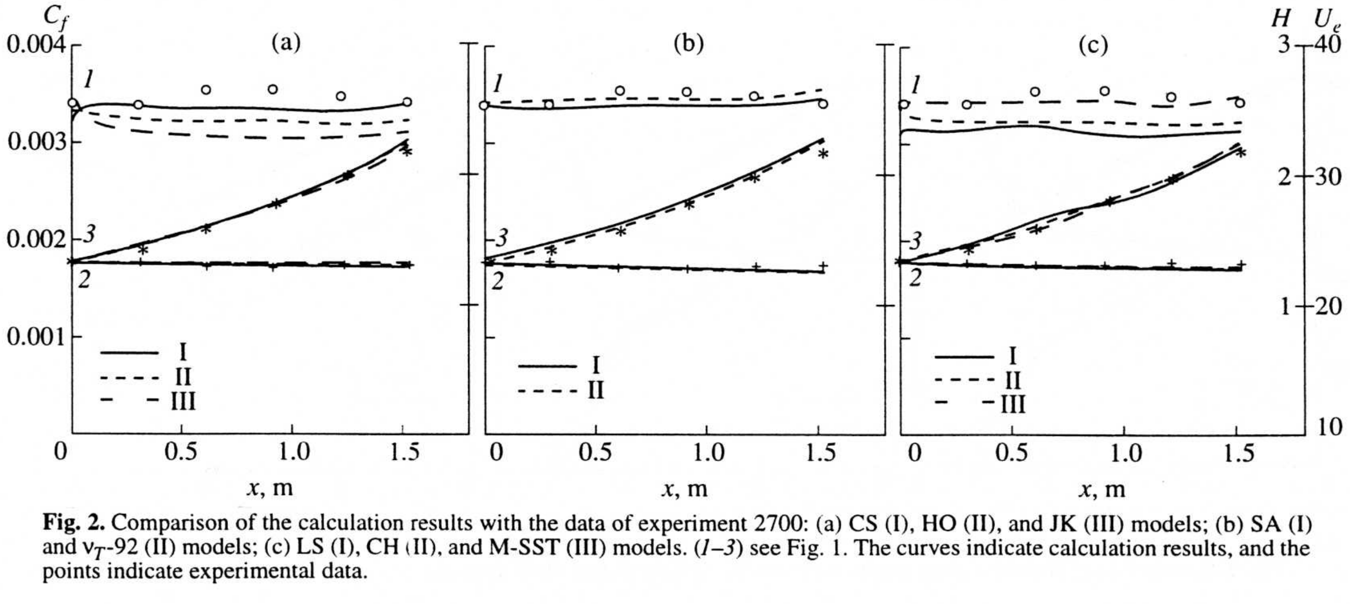

Fig. 2. Comparison of the calculation results with the data of experiment 2700: (a) CS (I), HO (II), and JK (III) models; (b) SA (I) and ν_T-92 (II) models; (c) LS (I), CH (II), and M-SST (III) models. (*1–3*) see Fig. 1. The curves indicate calculation results, and the points indicate experimental data.

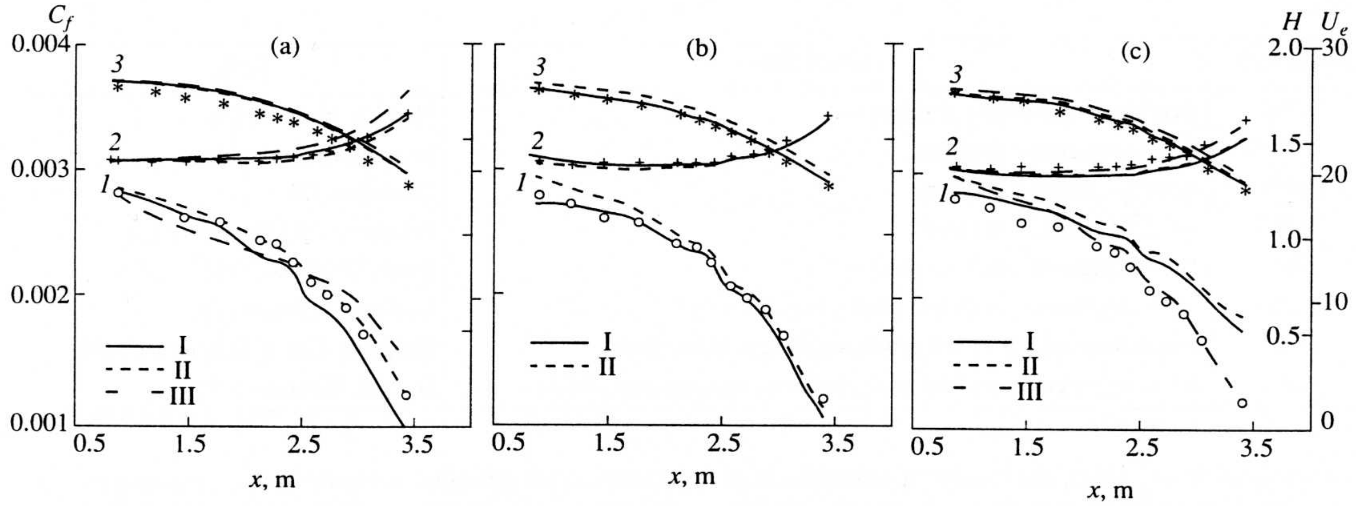

Fig. 3. Comparison of the calculation results with the data of Experiment 0141. Designations are the same as in Fig. 2.

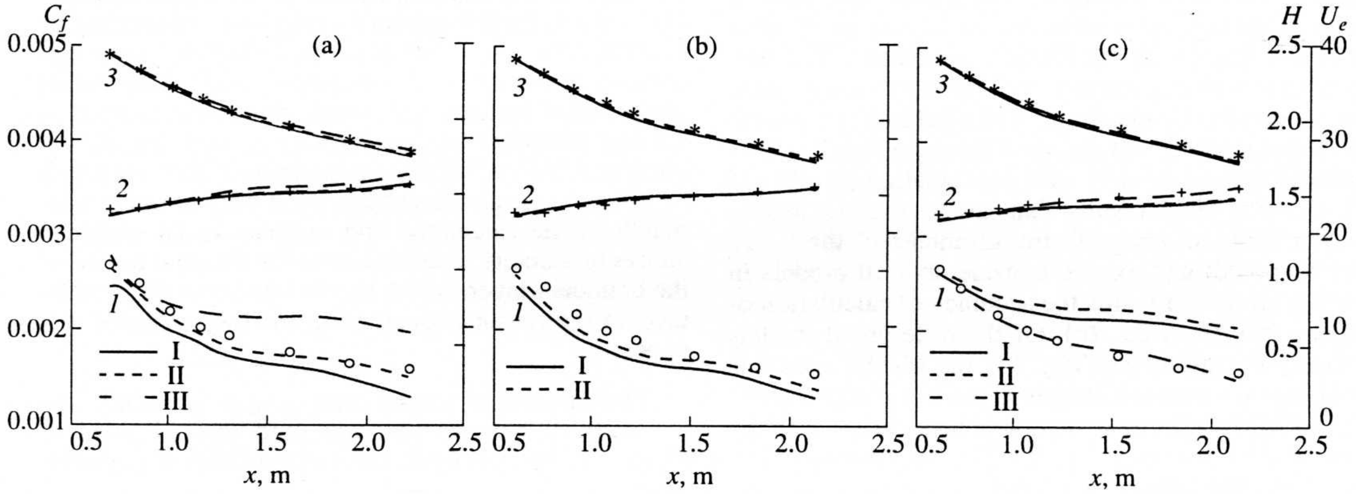

Fig. 4. Comparison of the calculation results with the data of Experiment 3300. Designations are the same as in Fig. 2.

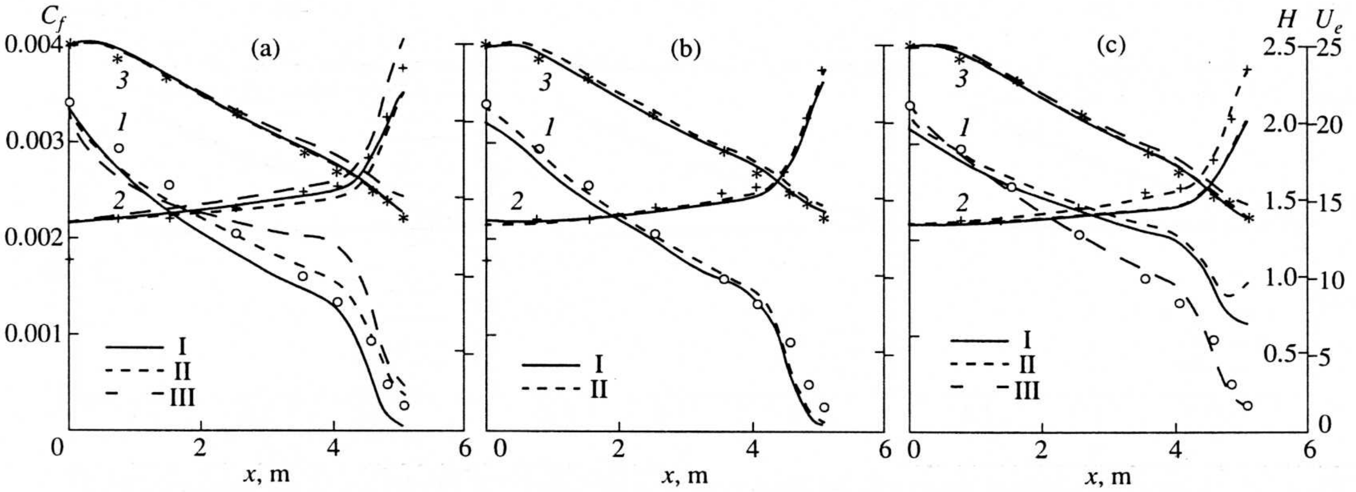

Fig. 5. Comparison of the calculation results with the data of Experiment 4800. Designations are the same as in Fig. 2.

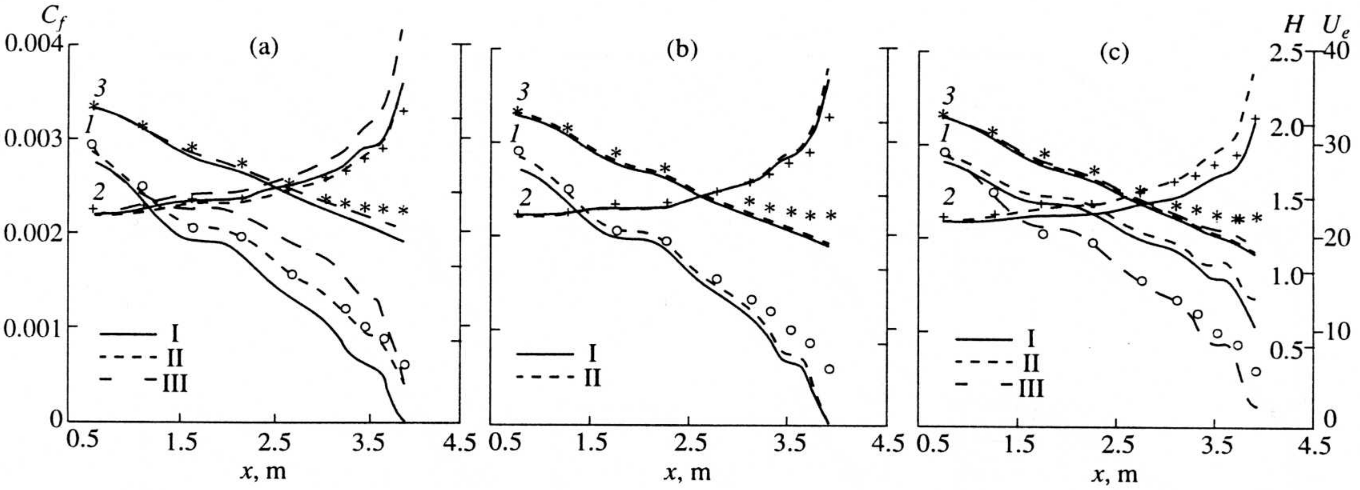

Fig. 6. Comparison of the calculation results with the data of Experiment 1200. Designations are the same as in Fig. 2.

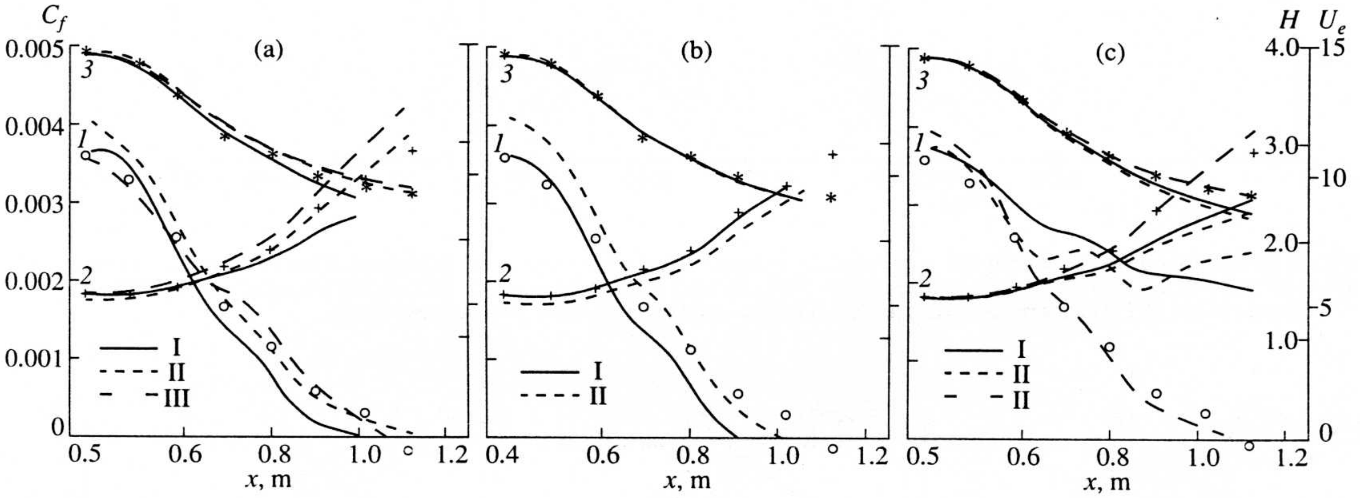

Fig. 7. Comparison of the calculation results with the data of Experiment DF. Designations are the same as in Fig. 2.

In all of the treated cases, the HO model is the most advantageous of the algebraic and semidifferential models (see Figs. 3a–7a). It provides for a fairly high accuracy of calculation of longitudinal distributions of all main characteristics of boundary layer in the entire region of flow, including in the vicinity of the separation layer, and enables one to calculate quite adequately the velocity profiles (Fig. 8) in this region, which presents most difficulties for simulation.

The characteristic drawback of the CS model is the underestimation of friction on the wall and, accordingly, the prediction of earlier (than in the experiments) separation of boundary layer. On the contrary, the semidifferential JK model first overestimates friction and then, on approaching separation, predicts its greater (than in the experiment) decrease.

The observed drawbacks of the CS and JK models also show up when calculating the velocity profiles in the boundary layer (Fig. 8) and, to a somewhat lesser degree, when calculating the longitudinal distribution of the form parameter of boundary layer $H = \delta^*/\theta$.

The results of calculations using two differential turbulence models with one equation (ν_T-92 and SA) being treated, as in the case of boundary layer with an accelerating pressure gradient, are rather close to one another (some exception being only the DF experiment, in which the axisymmetric flow was investigated rather than plane flow) and, by and large, agree adequately with experiment, though not as well as the HO model (see Figs. 3b–7b).

The analysis of the results derived from the calculation of boundary layers with an adverse pressure gradient using differential models of turbulence with two equations (Figs. 3c–7c) fully supports the conclusions, made as a result of numerous recent investigations (see, for example, [20, 21]), about the invalidity of models of

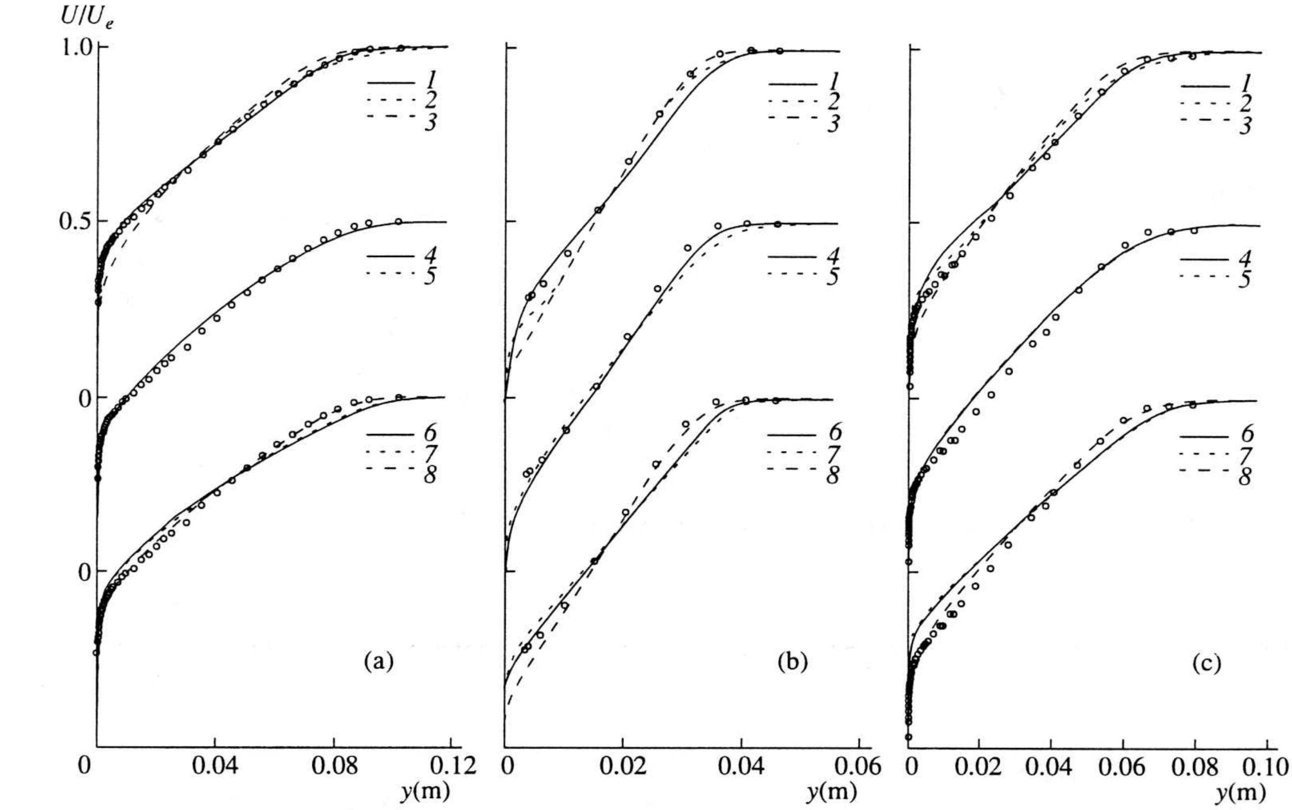

Fig. 8. Comparison of the calculated and experimentally obtained velocity profiles (a) for Experiment 0141 at $x = 3.4$ m, (b) for Experiment DF at $x = 0.931$ m, and (c) for Experiment 0431 at $x = 3.1$ m; *(1)* CS model, *(2)* HO, *(3)* JK, *(4)* SA, *(5)* ν_T-92, *(6)* LS, *(7)* CH, and *(8)* M-SST. The curves indicate calculation results, and the points indicate experimental data.

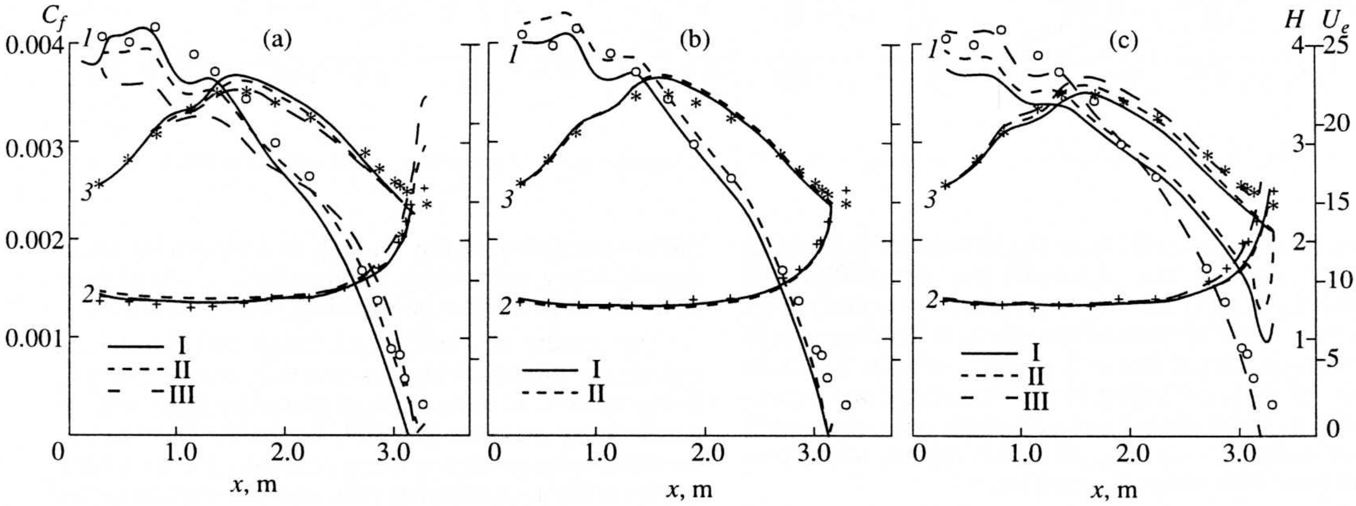

Fig. 9. Comparison of the calculation results with the data of Experiment 0431. Designations are the same as in Fig. 2.

the k–ε type for calculation of flow of this class. In all of the treated cases, models of this class (LS and CH models) greatly overestimate friction on the wall in the preseparation region and, in some cases (Experiment 4800, Fig. 5c), lead to a qualitatively incorrect behavior of the friction coefficient in the neighborhood of the separation point. As distinct from these models, the M-SST k–ω model in all of the treated cases yields fairly exact results with respect to all characteristics of boundary layers with an adverse pressure gradient and

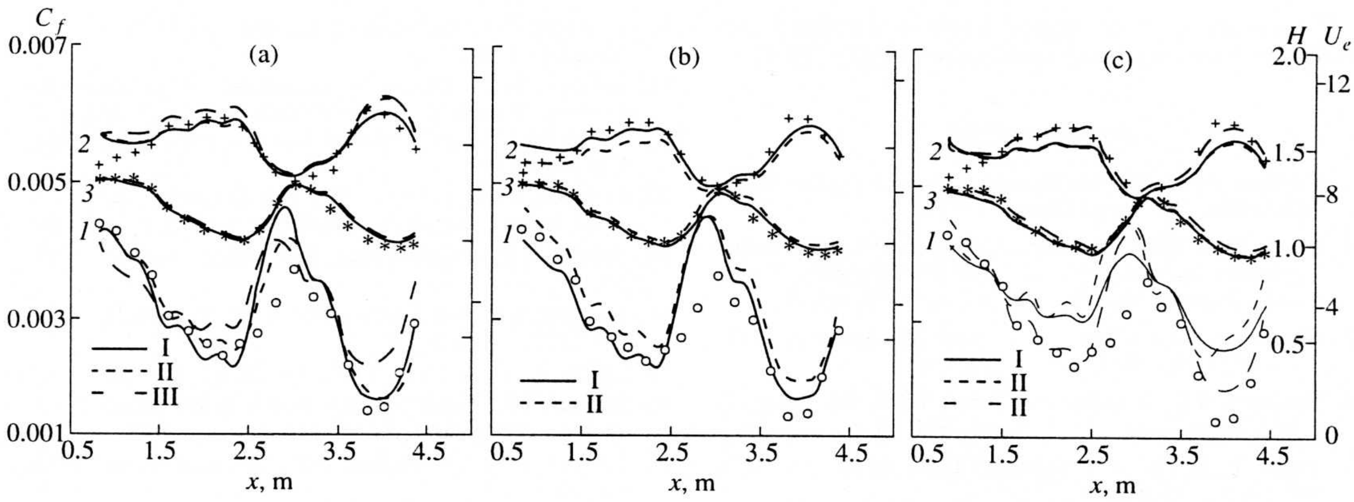

Fig. 10. Comparison of the calculation results with the data of Experiment TM. Designations are the same as in Fig. 2.

proves competitive with the HO model, which, for flows of this class, as was demonstrated above, is the best of the group of algebraic and semidifferential models.

We will now treat the results of calculations of a boundary layer with an alternating-sign pressure gradient (Experiments TM and 0431, Figs. 9 and 10). As was to be expected, these flows turned out to be most complicated for calculation, because, for their description, a turbulence model must respond adequately to rapid changes of the parameters of external flow, when significant importance is acquired by the "memory" effects that are little susceptible to simulation within semiempirical models based on turbulent viscosity. In particular, as is seen in Fig. 10, in this case even the best of the treated models (HO, ν_T-92, SA, and M-SST models), which enable one to fairly adequately calculate the characteristics of boundary layers with both accelerating and adverse pressure gradients, are incapable of describing with acceptable accuracy the reaction of flow to the change of gradient sign from positive (decelerated flow) to negative (accelerated flow). Although all models correctly describe this effect from the qualitative standpoint, the discrepancy between the calculation results and experiment in this region, for example, with respect to the friction coefficient, reaches 20–30%. On the other hand, these models, as distinct from the rest of the models, enable one to fairly accurately describe the variation of the characteristics of boundary layer during reverse variation of the acceleration sign (transition from accelerated to decelerated flow) in both the TM and the 0431 experiments.

CONCLUSION

In conclusion, we will briefly formulate the main results of our investigations.

It is shown that, in estimating the capabilities of the existing and calibrating new turbulence models by way of comparing the calculation results with experiment for turbulent boundary layers with a longitudinal pressure gradient, the inverse method of solving boundary layer equations proves more convenient and enables one to obtain more objective information about models than the traditional direct method. The inverse method was used to perform a detailed testing of a wide class of semiempirical turbulence models based on turbulent viscosity, as applied to the calculation of wall boundary layers with accelerating, adverse, and alternating-sign pressure gradients. It is shown that the Horton model (3) yields the best results from among algebraic and semidifferential turbulence models oriented to the calculation of flows of the type of boundary layer. Prominent among differential models are the Menter k–ω model and the ν_T-92 and Spalart-Allmaras models, which contain one differential equation for turbulent viscosity. They are all greatly superior to the models of the k–ε group (Launder–Sharma and Chien models) and quite competitive with the Horton model. In so doing, the Menter model yields better results for flows with a significant adverse pressure gradient than all other models. At the same time, none of the treated models provides a sufficiently accurate description of the characteristics of substantially nonequilibrium boundary layers with an alternating-sign pressure gradient in the region of transition from decelerated to accelerated flow.

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